

A Popov Stability Condition for Uncertain Linear Quantum Systems

Matthew R James, Ian R. Petersen and Valery Ugrinovskii

Abstract—This paper considers a Popov type approach to the problem of robust stability for a class of uncertain linear quantum systems subject to unknown perturbations in the system Hamiltonian. A general stability result is given for a general class of perturbations to the system Hamiltonian. Then, the special case of a nominal linear quantum system is considered with quadratic perturbations to the system Hamiltonian. In this case, a robust stability condition is given in terms of a frequency domain condition which is of the same form as the standard Popov stability condition.

I. INTRODUCTION

This paper builds on the previous papers [1]–[3] which consider the problem of robust stability analysis for open quantum systems subject to perturbations in either the system Hamiltonian or coupling operator, which together define the dynamics of the quantum system. The results of these papers can be regarded as extensions of the classical small gain theorem for robust stability to the case of quantum systems. The main contribution of this paper is a result which can be regarded as an extension of the classical Popov criterion for absolute stability to the case of open quantum systems. In particular, we extend the result of [2], in which the perturbations to the system Hamiltonian are uncertain quadratic perturbations, to obtain a corresponding Popov robust stability result.

The small gain theorem and the Popov criterion for absolute stability are two of the most useful tests for robust stability and nonlinear system stability; e.g., see [4]. Both of these stability tests consider a Lur’e system which is the feedback interconnection between a linear time invariant system and a sector bounded nonlinearity or uncertainty. The key distinction between the small gain theorem and the Popov criterion is that the small gain theorem establishes absolute stability via the use of a fixed quadratic Lyapunov

function whereas the Popov criterion relies on a Lyapunov function of the Lur’e Postnikov form which involves the sum of a quadratic term and a term dependent on the integral of the nonlinearity itself. The small gain theorem can be used to establish stability in the presence of time-varying uncertainties and nonlinearities whereas the Popov criterion only applies to static time-invariant nonlinearities. However, the Popov criterion is less conservative than the small gain theorem. Hence, we are motivated to obtain a quantum Popov stability criterion in order to obtain less conservative results.

The study of quantum feedback control theory has been the subject of increasing interest in recent years; e.g., see [5]–[17]. In particular, the papers [14], [18] consider a framework of quantum systems defined in terms of a triple (S, L, H) where S is a scattering matrix, L is a vector of coupling operators and H is a Hamiltonian operator. The paper [18] then introduces notions of dissipativity and stability for this class of quantum systems. As in the papers [1], [3], the results of this paper build on the stability results of [18] to obtain robust stability results for uncertain quantum systems in which the quantum system Hamiltonian is decomposed as $H = H_1 + H_2$ where H_1 is a known nominal Hamiltonian and H_2 is a perturbation Hamiltonian, which is contained in a specified set of Hamiltonians \mathcal{W} .

For this general class of uncertain quantum systems, the paper first obtains a general abstract version of the Popov stability criterion which requires finding a Lyapunov type operator to satisfy an operator inequality. The paper then considers the case in which the nominal Hamiltonian H_1 is a quadratic function of annihilation and creation operators and the coupling operator vector is a linear function of annihilation and creation operators. This case corresponds to a nominal linear quantum system; e.g., see [8], [9], [11], [12], [17]. Also, it is assumed that the perturbation Hamiltonian is quadratic but uncertain. In this special case, a robust stability stability criterion is obtained in terms of a frequency domain condition which takes the same form as the classical Popov stability criterion.

The remainder of the paper proceeds as follows. In Section II, we define the general class of uncertain quantum systems under consideration. In this section, we also present a general Popov type stability result for this class of quantum systems. In Section III, we consider a class of uncertain quadratic perturbation Hamiltonians. In Section IV, we specialize to the case of linear nominal quantum systems and obtain a robust stability result for this case in which the stability condition is a frequency domain condition in the same form as the classical Popov stability condition. In Section V we present an illustrative example involving a quantum system

This work was supported by the Australian Research Council (ARC) under projects DP110102322, DP1094650 and FL110100020, and the Air Force Office of Scientific Research (AFOSR). This material is based on research sponsored by the Air Force Research Laboratory, under agreement number FA2386-09-1-4089. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation thereon. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Research Laboratory or the U.S. Government.

Matthew R. James is with the ARC Centre for Quantum Computation and Communication Technology, Research School of Engineering, College of Engineering and Computer Science, The Australian National University, Canberra, ACT 0200, Australia. Email: Matthew.James@anu.edu.au.

Ian R. Petersen and Valery Ugrinovskii are with the School of Engineering and Information Technology, University of New South Wales at the Australian Defence Force Academy, Canberra ACT 2600, Australia. {i.r.petersen, v.ugrinovskii}@gmail.com. The work of Valery Ugrinovskii was carried out in part while he was a visitor at the Australian National University.

arising from an optical parametric amplifier. In Section VI, we present some conclusions.

II. QUANTUM SYSTEMS

We consider open quantum systems defined by parameters (S, L, H) where $H = H_1 + H_2$; e.g., see [14], [18]. The corresponding generator for this quantum system is given by

$$\mathcal{G}(X) = -i[X, H] + \mathcal{L}(X) \quad (1)$$

where $\mathcal{L}(X) = \frac{1}{2}L^*[X, L] + \frac{1}{2}[L^*, X]L$. Here, $[X, H] = XH - HX$ denotes the commutator between two operators and the notation $*$ denotes the adjoint of an operator. Also, H_1 is a self-adjoint operator on the underlying Hilbert space referred to as the nominal Hamiltonian and H_2 is a self-adjoint operator on the underlying Hilbert space referred to as the perturbation Hamiltonian. The triple (S, L, H) , along with the corresponding generators define the Heisenberg evolution $X(t)$ of an operator X according to a quantum stochastic differential equation; e.g., see [18].

The problem under consideration involves establishing robust stability properties for an uncertain open quantum system for the case in which the perturbation Hamiltonian is contained in a given set \mathcal{W}_1 . Using the notation of [18], the set \mathcal{W}_1 defines a set of exosystems. This situation is illustrated in the block diagram shown in Figure 1. The main

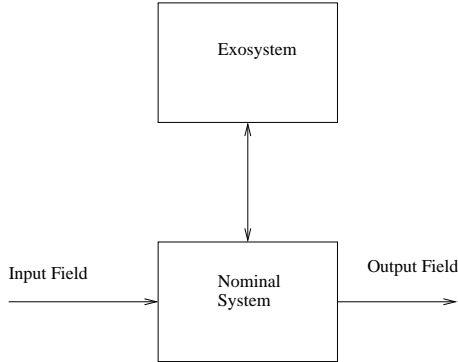


Fig. 1. Block diagram representation of an open quantum system interacting with an exosystem.

robust stability results presented in this paper will build on the following result from [18].

Lemma 1 (See Lemma 3.4 of [18].): Consider an open quantum system defined by (S, L, H) and suppose there exists a non-negative self-adjoint operator V on the underlying Hilbert space such that

$$\mathcal{G}(V) + cV \leq \lambda \quad (2)$$

where $c > 0$ and λ are real numbers. Then for any plant state, we have

$$\langle V(t) \rangle \leq e^{-ct} \langle V \rangle + \frac{\lambda}{c}, \quad \forall t \geq 0.$$

Here $V(t)$ denotes the Heisenberg evolution of the operator V and $\langle \cdot \rangle$ denotes quantum expectation; e.g., see [18].

We will also use following result, which is a slight modification of Theorem 3.1 and Lemma 3.4 of [18].

Lemma 2: Consider an open quantum system defined by (S, L, H) and suppose there exists non-negative self-adjoint operators V and W on the underlying Hilbert space such that the following *quantum dissipation inequality* holds

$$\mathcal{G}(V) + W \leq \lambda \quad (3)$$

where λ is a real number. Then for any plant state, we have

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle W(t) \rangle dt \leq \lambda. \quad (4)$$

Here $W(t)$ denotes the Heisenberg evolution of the operator W and $\langle \cdot \rangle$ denotes quantum expectation; e.g., see [18].

Proof The proof is similar to the proof of Lemma 3.4 in [18]. In a similar manner we obtain from (3)

$$\mathbb{E}_t \left[V(t+h) - V(t) + \int_t^{t+h} W(s) ds \right] \leq \lambda h. \quad (5)$$

where \mathbb{E}_t denotes vacuum expectation operator. Taking the vacuum expectation \mathbb{E}_0 on both sides, and noting that $\mathbb{E}_0 \mathbb{E}_t = \mathbb{E}_0$ results in

$$\begin{aligned} & \langle \psi, \mathbb{E}_0[V(t+h)]\psi \rangle - \langle \psi, \mathbb{E}_0[V(t)]\psi \rangle \\ & + \int_t^{t+h} \langle \psi, \mathbb{E}_0[W(s)]\psi \rangle ds \leq \lambda h \langle \psi, \psi \rangle \end{aligned}$$

for any ψ in the underlying Hilbert space. Then for any corresponding plant state (i.e., for ψ such that $\langle \psi, \psi \rangle = 1$) we have

$$\frac{d}{dt} \langle V(t) \rangle + \langle W(t) \rangle \leq \lambda.$$

Then (4) follows in a standard manner. \square

A. Commutator Decomposition

We now consider a set of self-adjoint perturbation Hamiltonians $H_2 \in \mathcal{W}_1$. For a given set of non-negative self-adjoint operators \mathcal{P} , a set of Popov scaling parameters $\Theta \subset [0, \infty)$, a self-adjoint operator H_1 , which is the nominal Hamiltonian, a coupling operator L , and for a real parameter $\beta \geq 0$, this set \mathcal{W}_1 is defined in terms of the commutator decompositions

$$\begin{aligned} [V - \theta H_1, H_2] &= [V - \theta H_1, z^\dagger]w - w^\dagger[z, V - \theta H_1], \\ \mathcal{L}(H_2) &\leq \mathcal{L}(z^\dagger)w + w^\dagger \mathcal{L}(z) + \beta[z, L]^\dagger[z, L] \end{aligned} \quad (6)$$

for all $V \in \mathcal{P}$ and $\theta \in \Theta$, where w and z are given operator vectors of the same dimension. Here, the notation † denotes the adjoint transpose of a vector of operators. In addition, the notation $\#$ denotes the vector of adjoint operators for a given vector of operators.

Then, the set \mathcal{W}_1 will be defined in terms of the sector bound condition

$$\left(w - \frac{1}{\gamma} z \right)^\dagger \left(w - \frac{1}{\gamma} z \right) \leq \frac{1}{\gamma^2} z^\dagger z \quad (7)$$

where $\gamma > 0$ is a given constant. That is, we define

$$\mathcal{W}_1 = \left\{ \begin{array}{l} H_2 \geq 0 : \exists w, z, \text{ such that (7)} \\ \text{and (6) are satisfied } \forall V \in \mathcal{P}, \theta \in \Theta \end{array} \right\}. \quad (8)$$

Using this definition, we obtain the following theorem.

Theorem 1: Consider a set of non-negative self-adjoint operators \mathcal{P} , an open quantum system (S, L, H) and an observable W where $H = H_1 + H_2$ and $H_2 \in \mathcal{W}_1$ defined in (8). Suppose there exists a $V \in \mathcal{P}$ and real constants $\theta \in \Theta$, $\tilde{\lambda} \geq 0$ such that

$$\begin{aligned} & -i[V, H_1] + \mathcal{L}(V) \\ & + \frac{1}{\gamma} (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z)^\dagger \\ & \times (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z) \\ & + \theta \beta[z, L]^\dagger [z, L] + W \leq \tilde{\lambda}. \end{aligned} \quad (9)$$

Then

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle W(t) \rangle dt \leq \tilde{\lambda}.$$

Here $W(t)$ denotes the Heisenberg evolution of the operator W .

Proof: Let $\theta \in \Theta$ and $\tilde{\lambda} \geq 0$ be given such that the conditions of the theorem are satisfied and consider $\mathcal{G}(V + \theta H_2)$ defined in (1). Then

$$\begin{aligned} \mathcal{G}(V + \theta H_2) &= -i[V + \theta H_2, H_1 + H_2] + \mathcal{L}(V + \theta H_2) \\ &= -i[V, H_1] - i\theta[H_2, H_1] - i[V, H_2] \\ &\quad - i\theta[H_2, H_2] + \mathcal{L}(V) + \theta \mathcal{L}(H_2) \\ &= -i[V, H_1] - i[V - \theta H_1, H_2] + \mathcal{L}(V) \\ &\quad + \theta \mathcal{L}(H_2). \end{aligned} \quad (10)$$

Using the decomposition in the first equation (6), we have

$$\begin{aligned} \mathcal{G}(V + \theta H_2) &= -i[V, H_1] + \mathcal{L}(V) - i[V - \theta H_1, z^\dagger]w \\ &\quad + iw^\dagger[z, V - \theta H_1] + \theta \mathcal{L}(H_2). \end{aligned} \quad (11)$$

Now

$$[V - \theta H_1, z^\dagger]^\dagger = z(V - \theta H_1) - (V - \theta H_1)z = [z, V - \theta H_1]$$

since $V - \theta H_1$ is self-adjoint. This confirms that the operator on the right hand side of the above identity is a self-adjoint operator. Therefore, the following inequality follows from the second equation (6):

$$\begin{aligned} \mathcal{G}(V + \theta H_2) &\leq -i[V, H_1] + \mathcal{L}(V) \\ &\quad - i[V - \theta H_1, z^\dagger]w + iw^\dagger[z, V - \theta H_1] \\ &\quad + \theta(\mathcal{L}(z^\dagger)w + w^\dagger \mathcal{L}(z)) + \theta \beta[z, L]^\dagger [z, L]. \end{aligned} \quad (12)$$

Also, note that

$$\begin{aligned} (\mathcal{L}(z))^\dagger &= \left(\frac{1}{2} L^\dagger [z, L] + \frac{1}{2} [L^\dagger, z] L \right)^\dagger \\ &= \frac{1}{2} [L^\dagger, z^\dagger] L + \frac{1}{2} L^\dagger [z^\dagger, L] \\ &= \mathcal{L}(z^\dagger). \end{aligned} \quad (13)$$

Furthermore,

$$\begin{aligned} 0 &\leq \left(\frac{i[z, V - \theta H_1] + \theta \mathcal{L}(z)}{\sqrt{\gamma}} - \sqrt{\gamma}(w - z/\gamma) \right)^\dagger \\ &\quad \times \left(\frac{i[z, V - \theta H_1] + \theta \mathcal{L}(z)}{\sqrt{\gamma}} - \sqrt{\gamma}(w - z/\gamma) \right) \\ &= \frac{1}{\gamma} (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z)^\dagger \\ &\quad \times (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z) \\ &\quad - (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z)^\dagger w \\ &\quad - w^\dagger (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z) + \gamma w^\dagger w. \end{aligned}$$

This implies that

$$\begin{aligned} & (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z)^\dagger w \\ & + w^\dagger (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z) - \gamma w^\dagger w \\ & \leq \frac{1}{\gamma} (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z)^\dagger \\ & \quad \times (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z). \end{aligned}$$

Using this inequality and (12), we have

$$\begin{aligned} & \mathcal{G}(V + \theta H_2) + \frac{1}{\gamma} z^\dagger z - \gamma \left(w - \frac{1}{\gamma} z \right)^\dagger \left(w - \frac{1}{\gamma} z \right) \\ & \leq -i[V, H_1] + \mathcal{L}(V) \\ & \quad + (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z)^\dagger w \\ & \quad + w^\dagger (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z) \\ & \quad - \gamma w^\dagger w + \theta \beta[z, L]^\dagger [z, L] \\ & \leq -i[V, H_1] + \mathcal{L}(V) \\ & \quad + \frac{1}{\gamma} (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z)^\dagger \\ & \quad \times (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z) \\ & \quad + \theta \beta[z, L]^\dagger [z, L]. \end{aligned} \quad (14)$$

Then it follows from (7) and (9) that

$$\mathcal{G}(V + \theta H_2) + W \leq \tilde{\lambda}.$$

The result of the theorem then follows from Lemma 2. \square

III. QUADRATIC PERTURBATIONS OF THE HAMILTONIAN

We consider a set \mathcal{W}_2 of quadratic perturbation Hamiltonians of the form

$$H_2 = \frac{1}{2} \begin{bmatrix} \zeta^\dagger & \zeta^T \end{bmatrix} \Delta \begin{bmatrix} \zeta \\ \zeta^\# \end{bmatrix} \quad (15)$$

where $\Delta \in \mathbb{C}^{2m \times 2m}$ is a Hermitian matrix of the form

$$\Delta = \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2^\# & \Delta_1^\# \end{bmatrix}, \quad (16)$$

$\Delta_1 = \Delta_1^\dagger$, $\Delta_2 = \Delta_2^T$ and ζ is a given vector of operators. Here, in the case of complex matrices, the notation † refers to the complex conjugate transpose of a matrix. Also, in the case of complex matrices, the notation $^\#$ refers to the complex conjugate matrix. In addition, for this case we assume that $\Theta = [0, \infty)$.

The matrix Δ is subject to the bounds

$$0 \leq \Delta \leq \frac{4}{\gamma} I. \quad (17)$$

Then we define

$$\mathcal{W}_2 = \left\{ \begin{array}{l} H_2 \text{ of the form (15), (16) such that} \\ \text{condition (17) is satisfied} \end{array} \right\}. \quad (18)$$

Using this definition, we obtain the following lemma.

Lemma 3: Suppose that $[z, L]$ is a constant vector. Then, for any set of self-adjoint operators \mathcal{P} ,

$$\mathcal{W}_2 \subset \mathcal{W}_1.$$

Proof: Given any $H_2 \in \mathcal{W}_2$, let $z = \begin{bmatrix} \zeta \\ \zeta^\# \end{bmatrix}$ and $w = \frac{1}{2}\Delta z$.

Hence, $H_2 = w^\dagger z$. Then, for any $\tilde{V} \in \mathcal{P}$ and $\theta \geq 0$, let $\tilde{V} = V - \theta H_1$ and we have

$$\begin{aligned} [\tilde{V}, z^\dagger]w &= \frac{1}{2} \left(\tilde{V}z^\dagger - z^\dagger \tilde{V} \right) \Delta z \\ &= \frac{1}{2} \tilde{V}z^\dagger \Delta z - \frac{1}{2} z^\dagger \Delta \tilde{V}z, \end{aligned}$$

since \tilde{V} is a scalar operator and Δ is a constant matrix. Also,

$$\begin{aligned} w^\dagger[z, \tilde{V}] &= \frac{1}{2} z^\dagger \Delta (z\tilde{V} - \tilde{V}z) \\ &= \frac{1}{2} z^\dagger \Delta z \tilde{V} - \frac{1}{2} z^\dagger \Delta \tilde{V}z. \end{aligned}$$

Hence,

$$\begin{aligned} [\tilde{V}, z^\dagger]w - w^\dagger[z, \tilde{V}] &= \frac{1}{2} \tilde{V}z^\dagger \Delta z - \frac{1}{2} z^\dagger \Delta z \tilde{V} \\ &= \frac{1}{2} [\tilde{V}, z^\dagger \Delta z] \\ &= [\tilde{V}, H_2]. \end{aligned}$$

Similarly,

$$[L, z^\dagger]w - w^\dagger[z, L] = [L, H_2]. \quad (19)$$

In addition,

$$\begin{aligned} w^T[z, L]^\# L - L[z, L]^\dagger w &= \frac{1}{2} z^T \Delta^T [z, L]^\# L - \frac{1}{2} L[z, L]^\dagger \Delta z \\ &= \frac{1}{2} [z, L]^\dagger \Delta z L - \frac{1}{2} L[z, L]^\dagger \Delta L z \\ &= \frac{1}{2} [z, L]^\dagger \Delta [z, L] \\ &= \frac{1}{2} [z, L]^\dagger \Delta [z, L] \end{aligned} \quad (20)$$

and similarly

$$L^*[z, L]^T w^\# - w^\dagger[z, L]L^* = \frac{1}{2} [z, L]^\dagger \Delta [z, L]. \quad (21)$$

Now using (19), (20), (21) and the assumption that $[z, L]$ is a constant vector, it follows that

$$\begin{aligned} \mathcal{L}(H_2) &= \frac{1}{2} L^*[H_2, L] + \frac{1}{2} [L^*, H_2]L \\ &= -\frac{1}{2} L^* ([L, z^\dagger]w - w^\dagger[z, L]) \\ &\quad - \frac{1}{2} (w^\dagger[L, z^\dagger]^\dagger - [z, L]^\dagger w) L \\ &= \frac{1}{2} L^*[z^\dagger, L]w + \frac{1}{2} L^* w^\dagger[z, L] \\ &\quad + \frac{1}{2} w^\dagger[z^\dagger, L]^\dagger L + \frac{1}{2} [z, L]^\dagger w L \\ &= \frac{1}{2} L^*[z^\dagger, L]w + \frac{1}{2} L^*[z, L]^T w^\# \\ &\quad + \frac{1}{2} w^\dagger[z^\dagger, L]^\dagger L + \frac{1}{2} w^T[z, L]^\# L \\ &= \frac{1}{2} L^*[z^\dagger, L]w + \frac{1}{2} w^\dagger[z, L]L^* \\ &\quad + \frac{1}{2} w^\dagger[z^\dagger, L]^\dagger L + \frac{1}{2} L[z, L]^\dagger w \\ &\quad + [z, L]^\dagger \Delta [z, L] \\ &= \frac{1}{2} L^*[z^\dagger, L]w + \frac{1}{2} [L^*, z^\dagger]Lw \\ &\quad + \frac{1}{2} w^\dagger L^*[z, L] + \frac{1}{2} w^\dagger[L^*, z]L \\ &\quad + [z, L]^\dagger \Delta [z, L] \\ &= \mathcal{L}(z^\dagger)w + w^\dagger \mathcal{L}(z) + [z, L]^\dagger \Delta [z, L]. \end{aligned}$$

It then follows from (17) that

$$\mathcal{L}(H_2) \leq \mathcal{L}(z^\dagger)w + w^\dagger \mathcal{L}(z) + \frac{4}{\gamma} [z, L]^\dagger [z, L].$$

Therefore we can conclude that both of the conditions in (6) are satisfied with $\beta = \frac{4}{\gamma}$. Also, condition (17) implies

$$H_2 = w^\dagger z = \frac{1}{2} z^\dagger \Delta z \geq 0,$$

and

$$\begin{aligned} \left(w - \frac{1}{\gamma} z \right)^\dagger \left(w - \frac{1}{\gamma} z \right) &= w^\dagger w - \frac{1}{\gamma} z^\dagger w \\ &\quad - \frac{1}{\gamma} w^\dagger z + \frac{1}{\gamma^2} z^\dagger z \\ &= \frac{1}{4} z^\dagger \Delta \Delta z - \frac{1}{\gamma} z^\dagger \Delta z \\ &\quad + \frac{1}{\gamma^2} z^\dagger z \\ &\leq \frac{1}{\gamma^2} z^\dagger z \end{aligned}$$

which implies (7). Hence, $H_2 \in \mathcal{W}_1$. Therefore, $\mathcal{W}_2 \subset \mathcal{W}_1$. \square

IV. THE LINEAR CASE

We now consider the case in which the nominal quantum system corresponds to a linear quantum system; e.g., see [8], [9], [11], [12], [17]. In this case, we assume that H_1 is of the form

$$H_1 = \frac{1}{2} \begin{bmatrix} a^\dagger & a^T \end{bmatrix} M \begin{bmatrix} a \\ a^\# \end{bmatrix} \quad (22)$$

where $M \in \mathbb{C}^{2n \times 2n}$ is a Hermitian matrix of the form

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^\# & M_1^\# \end{bmatrix}$$

and $M_1 = M_1^\dagger$, $M_2 = M_2^T$. Here a is a vector of annihilation operators on the underlying Hilbert space and $a^\#$ is the corresponding vector of creation operators. The annihilation and creation operators are assumed to satisfy the canonical commutation relations:

$$\begin{aligned} \left[\begin{bmatrix} a \\ a^\# \end{bmatrix}, \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \right] &= \begin{bmatrix} a \\ a^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \\ &\quad - \left(\begin{bmatrix} a \\ a^\# \end{bmatrix}^\# \begin{bmatrix} a \\ a^\# \end{bmatrix}^T \right)^T \\ &= J \end{aligned} \quad (23)$$

where $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$; e.g., see [10], [15], [17].

In addition, we assume L is of the form

$$L = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} = \tilde{N} \begin{bmatrix} a \\ a^\# \end{bmatrix} \quad (24)$$

where $N_1 \in \mathbb{C}^{1 \times n}$ and $N_2 \in \mathbb{C}^{1 \times n}$. Also, we write

$$\begin{bmatrix} L \\ L^\# \end{bmatrix} = N \begin{bmatrix} a \\ a^\# \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \\ N_2^\# & N_1^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

In addition, we assume that V is of the form

$$V = \begin{bmatrix} a^\dagger & a^T \end{bmatrix} P \begin{bmatrix} a \\ a^\# \end{bmatrix} \quad (25)$$

where $P \in \mathbb{C}^{2n \times 2n}$ is a positive-definite Hermitian matrix of the form

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^\# & P_1^\# \end{bmatrix}. \quad (26)$$

Hence, we consider the set of non-negative self-adjoint operators \mathcal{P}_1 defined as

$$\mathcal{P}_1 = \left\{ \begin{array}{l} V \text{ of the form (25) such that } P > 0 \text{ is a} \\ \text{Hermitian matrix of the form (26)} \end{array} \right\}. \quad (27)$$

In the linear case, we also let $\zeta = E_1 a + E_2 a^\#$ and hence we can write

$$z = \begin{bmatrix} \zeta \\ \zeta^\# \end{bmatrix} = \begin{bmatrix} E_1 & E_2 \\ E_2^\# & E_1^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} = E \begin{bmatrix} a \\ a^\# \end{bmatrix}. \quad (28)$$

We will also consider a specific notion of robust mean square stability.

Definition 1: An uncertain open quantum system defined by (S, L, H) where $H = H_1 + H_2$ with H_1 of the form (22), $H_2 \in \mathcal{W}$, and L of the form (24) is said to be *robustly mean square stable* if for any $H_2 \in \mathcal{W}$, there exist constants $c_1 > 0$, $c_2 > 0$ and $c_3 \geq 0$ such that

$$\begin{aligned} &\left\langle \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix}^\dagger \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix} \right\rangle \\ &\leq c_1 e^{-c_2 t} \left\langle \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \begin{bmatrix} a \\ a^\# \end{bmatrix} \right\rangle + c_3 \quad \forall t \geq 0. \end{aligned} \quad (29)$$

Here $\begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix}$ denotes the Heisenberg evolution of the vector of operators $\begin{bmatrix} a \\ a^\# \end{bmatrix}$; e.g., see [18].

In order to address the issue of robust mean square stability for the uncertain linear quantum systems under consideration, we first require some algebraic identities.

Lemma 4: Given $V \in \mathcal{P}_1$, H_1 defined as in (22) and L defined as in (24), then

$$\begin{aligned} [V, H_1] &= \\ &\quad \left[\begin{bmatrix} a^\dagger & a^T \end{bmatrix} P \begin{bmatrix} a \\ a^\# \end{bmatrix}, \frac{1}{2} \begin{bmatrix} a^\dagger & a^T \end{bmatrix} M \begin{bmatrix} a \\ a^\# \end{bmatrix} \right] \\ &= \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger [PJM - MJ P] \begin{bmatrix} a \\ a^\# \end{bmatrix}. \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{L}(V) &= \\ &\quad \frac{1}{2} L^\dagger [V, L] + \frac{1}{2} [L^\dagger, V] L \\ &= \text{Tr} \left(P J N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N J \right) \\ &\quad - \frac{1}{2} \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger (N^\dagger J N J P + P J N^\dagger J N) \begin{bmatrix} a \\ a^\# \end{bmatrix}. \end{aligned}$$

Proof: The proof of these identities follows via straightforward but tedious calculations using (23). \square

Lemma 5: With the variable z defined as in (28) and L defined as in (24), then

$$[z, L] = \begin{bmatrix} E \begin{bmatrix} a \\ a^\# \end{bmatrix}, \tilde{N} \begin{bmatrix} a \\ a^\# \end{bmatrix} \end{bmatrix} = E J \Sigma \tilde{N}^T$$

which is a constant vector. Here,

$$\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Proof: The proof of this result follows via straightforward but tedious calculations using (23). \square

Lemma 6: With the variable z defined as in (28), H_1 defined in (22) and L defined as in (24), then

$$\begin{aligned} -i[z, H_1] + \mathcal{L}(z) &= E \left(-iJM - \frac{1}{2} J N^\dagger J N \right) \begin{bmatrix} a \\ a^\# \end{bmatrix} \\ &= EA \begin{bmatrix} a \\ a^\# \end{bmatrix} \end{aligned}$$

where

$$A = -iJM - \frac{1}{2} J N^\dagger J N. \quad (30)$$

Furthermore,

$$i[z, V] = 2iEJP \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

Proof: The proofs of these equations follows via straightforward but tedious calculations using (23). \square

We will show that a sufficient condition for robust mean square stability when $H_2 \in \mathcal{W}_2$ is the existence of a constant $\theta \geq 0$, such that the following conditions are satisfied:

(i) The matrix A defined in (30) is Hurwitz.

(ii) The transfer function

$$G(s) = -2iE(sI - A)^{-1}JE^\dagger \quad (31)$$

satisfies the strict positive real (SPR) condition

$$\gamma I - (1 + \theta i\omega)G(i\omega) - (1 - \theta i\omega)G(i\omega)^\dagger > 0 \quad (32)$$

for all $\omega \in [-\infty, \infty]$.

This leads to the following theorem.

Theorem 2: Consider an uncertain open quantum system defined by (S, L, H) such that $H = H_1 + H_2$ where H_1 is of the form (22), L is of the form (24) and $H_2 \in \mathcal{W}_2$. Furthermore, assume that there exist a constant $\theta \geq 0$ such that the matrix A defined in (30) is Hurwitz and the frequency domain condition (32) is satisfied. Then the uncertain quantum system is robustly mean square stable.

Proof of Theorem 2. If the conditions of the theorem are satisfied, then the transfer function $\frac{\gamma}{2}I - (1 + \theta s)G(s)$ is strictly positive real. However, this transfer function has a state space realization

$$\frac{\gamma}{2}I - (1 + \theta s)G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline -C - \theta CA & \frac{\gamma}{2}I - \theta CB \end{array} \right]$$

where A is defined as in (30),

$$B = -2iJE^\dagger \quad (33)$$

and

$$C = E. \quad (34)$$

It now follows using the strict positive real lemma that the linear matrix inequality

$$\left[\begin{array}{cc} PA + A^\dagger P & PB + C^\dagger + \theta A^\dagger C^\dagger \\ B^\dagger P + C + \theta CA & -\gamma I + \theta(CB + B^\dagger C^\dagger) \end{array} \right] < 0 \quad (35)$$

will have a solution $P > 0$ of the form (26); e.g., see [4]. This matrix P defines a corresponding operator $V \in \mathcal{P}_1$ as in (25). Furthermore, it is straightforward to verify that $CB + B^\dagger C^\dagger = 0$. Hence, using Schur complements, it follows from (35) that

$$PA + A^\dagger P + \frac{1}{\gamma}(PB + C^\dagger + \theta A^\dagger C^\dagger)(B^\dagger P + C + \theta CA) < 0. \quad (36)$$

Now using Lemma 6 we have

$$\begin{aligned} i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z \\ &= i[z, V] + \theta(-i[z, H_1] + \mathcal{L}(z)) + z \\ &= (2iEJP + \theta EA + E) \begin{bmatrix} a \\ a^\# \end{bmatrix} \\ &= E(2iJP + \theta A + I) \begin{bmatrix} a \\ a^\# \end{bmatrix}. \end{aligned}$$

Hence using Lemma 4, we obtain

$$\begin{aligned} &-i[V, H_1] + \mathcal{L}(V) \\ &+ \frac{1}{\gamma}(i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z)^\dagger \\ &\times (i[z, V - \theta H_1] + \theta \mathcal{L}(z) + z) \\ &+ \frac{4\theta}{\gamma}[z, L]^\dagger[z, L] \\ &= \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \tilde{M} \begin{bmatrix} a \\ a^\# \end{bmatrix} \\ &+ \text{Tr} \left(P J N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N J \right) \\ &+ \frac{4\theta}{\gamma} \tilde{N}^\# \Sigma J E^\dagger E J \Sigma \tilde{N}^T \end{aligned} \quad (37)$$

where

$$\begin{aligned} \tilde{M} &= PA + A^\dagger P + \\ &\frac{1}{\gamma}(2iJP + \theta A + I)^\dagger E^\dagger E (2iJP + \theta A + I) \\ &= PA + A^\dagger P \\ &+ \frac{1}{\gamma}(PB + C^\dagger + \theta A^\dagger C^\dagger)(B^\dagger P + C + \theta CA), \end{aligned}$$

A is defined in (30), B is defined in (33) and C is defined in (34). From this, it follows using Lemma 5, Lemma 3, (36), and a similar argument to the proof of Theorem 1 that

$$\mathcal{G}(V + \theta H_2) \leq \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \tilde{M} \begin{bmatrix} a \\ a^\# \end{bmatrix} + \lambda$$

where

$$\begin{aligned} \lambda &= \text{Tr} \left(P J N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N J \right) \\ &+ \frac{4\theta}{\gamma} \tilde{N}^\# \Sigma J E^\dagger E J \Sigma \tilde{N}^T \\ &\geq 0. \end{aligned}$$

It follows from (36) that $\tilde{M} < 0$. Hence using (17), it follows that there exists a constant $c > 0$ such that the condition

$$\mathcal{G}(V + \theta H_2) + c(V + \theta H_2) \leq \lambda$$

is satisfied. Therefore, it follows from Lemma 1, Lemma 3, (17) and $P > 0$ that

$$\begin{aligned} &\left\langle \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix}^\dagger \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix} \right\rangle \leq \\ &e^{-ct} \left\langle \begin{bmatrix} a(0) \\ a^\#(0) \end{bmatrix}^\dagger \begin{bmatrix} a(0) \\ a^\#(0) \end{bmatrix} \right\rangle \frac{\lambda_{\max}[P + \frac{4\theta}{\gamma}E^\dagger E]}{\lambda_{\min}[P]} \\ &+ \frac{\lambda}{c\lambda_{\min}[P]} \quad \forall t \geq 0. \end{aligned} \quad (38)$$

Hence, the condition (29) is satisfied with $c_1 = \frac{\lambda_{\max}[P + \frac{4\theta}{\gamma}E^\dagger E]}{\lambda_{\min}[P]} > 0$, $c_2 = c > 0$ and $c_3 = \frac{\lambda}{c\lambda_{\min}[P]} \geq 0$. \square

Observation 1: A useful special case of the above result occurs when the QSDEs describing the nominal open quantum linear system depend only on annihilation operators and

not on the creation operators; e.g., see [11], [12]. This case corresponds to the case of $M_2 = 0$ and $N_2 = 0$. Also, we assume that $E_2 = 0$. In this case, we calculate the matrix A in (30) to be

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_1^\# \end{bmatrix}$$

where $A_1 = -iM_1 - \frac{1}{2}N_1^\dagger N_1$. Also, we calculate the transfer function matrix $G(s)$ in (31) to be

$$G(s) = -2 \begin{bmatrix} G_1(s) & 0 \\ 0 & -G_1(s^*)^\# \end{bmatrix}$$

where $G_1(s) = iE_1(sI - A_1)^{-1}E_1^\dagger$.

We now consider the case in which $G_1(s)$ is a SISO transfer function. In this case, the condition that the matrix A in (30) is Hurwitz reduces to the condition that the matrix

$$A_1 = -iM_1 - \frac{1}{2}N_1^\dagger N_1 \quad (39)$$

is Hurwitz. Also, the the SPR condition (32) reduces to the following conditions:

$$\frac{\gamma}{4} + \operatorname{Re}[G_1(i\omega)] - \theta\omega\operatorname{Im}[G_1(i\omega)] > 0; \quad (40)$$

$$\frac{\gamma}{4} - \operatorname{Re}[G_1(i\omega)] + \theta\omega\operatorname{Im}[G_1(i\omega)] > 0 \quad (41)$$

for all $\omega \in [-\infty, \infty]$. The conditions (40), (41) can be tested graphically producing a plot of $\omega\operatorname{Re}[G_1(i\omega)]$ versus $\operatorname{Im}[G_1(i\omega)]$ with $\omega \in [-\infty, \infty]$ as a parameter. Such a parametric plot is referred to as the Popov plot; e.g., see [4]. Then, the conditions (40), (41) will be satisfied if and only if the Popov plot lies between two straight lines of slope $\frac{1}{\theta}$ and with x -axis intercepts $\pm\frac{\gamma}{4}$; see Figure 2.

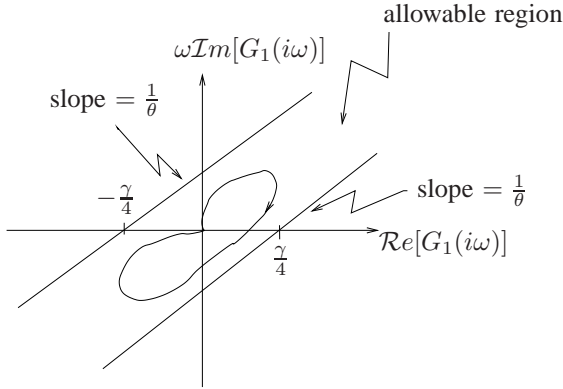


Fig. 2. Allowable region for the Popov plot.

V. ILLUSTRATIVE EXAMPLE

In this section, we consider an example of an open quantum system with

$$S = I, \quad H = \frac{1}{2}i((a^*)^2 - a^2), \quad L = \sqrt{\kappa}a,$$

which corresponds an optical parametric amplifier; see [19]. This is the same example which was considered in [2]. In order apply the theory of this paper to this example, we let

$$H_2 = \frac{1}{2} \begin{bmatrix} a^* & a \end{bmatrix} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix} \geq 0$$

and

$$H_1 = \frac{1}{2} \begin{bmatrix} a^* & a \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix}$$

so that $H_1 + H_2 = H$. This defines a linear quantum system of the form considered in Theorem 2 with $M_1 = -1$, $M_2 = 0$, $N_1 = \sqrt{\kappa}$, $N_2 = 0$, $E_1 = 1$, $E_2 = 0$. Also, $\Delta = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \geq 0$, which satisfies $\Delta \leq 2I$. Hence, we can choose $\gamma = 2$ to ensure that condition (17) is satisfied and therefore $H_2 \in \mathcal{W}_2$. Also, note that this system is a system of the form considered in Observation 1 with $A_1 = i - \frac{\kappa}{2}$, which is Hurwitz for all $\kappa > 0$, and $G_1(s) = \frac{i}{s - i + \frac{\kappa}{2}}$. We then choose $\kappa = 2.1$ and construct the Popov plot corresponding to the transfer function $G_1(s)$ as discussed in Observation 1. For a value of $\theta = 0.2$, this plot, along with the corresponding allowable region, is shown in Figure 3. From this figure it can be seen that the Popov plot lies within the allowable region and hence, it follows from Theorem 2 and Observation 1 that this system will be mean square stable for $\kappa = 2.1$. However, the method of [2] could only prove that this system was stable for $\kappa > 4$. Hence for this example, the proposed method provides a considerable improvement over the method of [2].

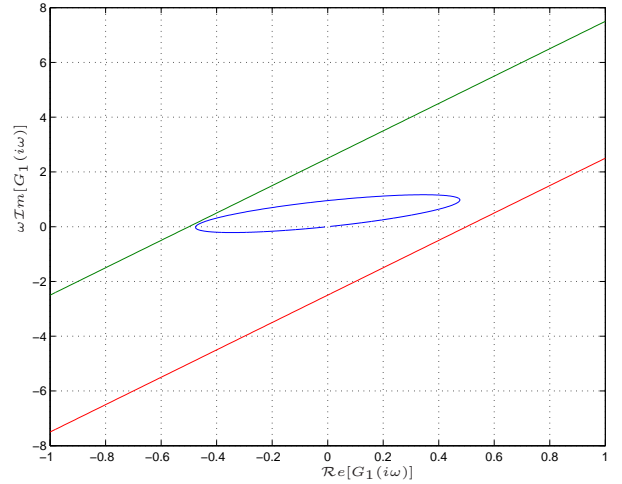


Fig. 3. Popov plot for the optical parametric amplifier system.

VI. CONCLUSIONS

In this paper, we have considered the problem of robust stability for uncertain linear quantum systems with uncertain quadratic perturbations to the system Hamiltonian. The stability condition which is obtained is a quantum version of the classical Popov stability criterion. This frequency domain condition is less conservative than a previous stability result obtained which takes the form of a quantum version of the classical small gain theorem.

REFERENCES

- [1] I. R. Petersen, V. Ugrinovskii, and M. R. James, "Robust stability of uncertain quantum systems," in *Proceedings of the 2012 American Control Conference*, Montreal, Canada, June 2012.

- [2] —, “Robust stability of uncertain linear quantum systems,” *Philosophical Transactions of the Royal Society A*, 2012, to appear, Accepted 3 April 2012.
- [3] —, “Robust stability of quantum systems with a nonlinear coupling operator,” in *Proceedings of the 51st IEEE Conference on Decision and Control*, Maui, December 2012, to appear, accepted 18/7/2012.
- [4] H. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.
- [5] M. Yanagisawa and H. Kimura, “Transfer function approach to quantum control-part I: Dynamics of quantum feedback systems,” *IEEE Transactions on Automatic Control*, vol. 48, no. 12, pp. 2107–2120, 2003.
- [6] —, “Transfer function approach to quantum control-part II: Control concepts and applications,” *IEEE Transactions on Automatic Control*, vol. 48, no. 12, pp. 2121–2132, 2003.
- [7] N. Yamamoto, “Robust observer for uncertain linear quantum systems,” *Phys. Rev. A*, vol. 74, pp. 032 107–1 – 032 107–10, 2006.
- [8] M. R. James, H. I. Nurdin, and I. R. Petersen, “ H^∞ control of linear quantum stochastic systems,” *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1787–1803, 2008.
- [9] H. I. Nurdin, M. R. James, and I. R. Petersen, “Coherent quantum LQG control,” *Automatica*, vol. 45, no. 8, pp. 1837–1846, 2009.
- [10] J. Gough, R. Gohm, and M. Yanagisawa, “Linear quantum feedback networks,” *Physical Review A*, vol. 78, p. 062104, 2008.
- [11] A. I. Maalouf and I. R. Petersen, “Bounded real properties for a class of linear complex quantum systems,” *IEEE Transactions on Automatic Control*, vol. 56, no. 4, pp. 786 – 801, 2011.
- [12] —, “Coherent H^∞ control for a class of linear complex quantum systems,” *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 309–319, 2011.
- [13] N. Yamamoto, H. I. Nurdin, M. R. James, and I. R. Petersen, “Avoiding entanglement sudden-death via feedback control in a quantum network,” *Physical Review A*, vol. 78, no. 4, p. 042339, 2008.
- [14] J. Gough and M. R. James, “The series product and its application to quantum feedforward and feedback networks,” *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2530–2544, 2009.
- [15] J. E. Gough, M. R. James, and H. I. Nurdin, “Squeezing components in linear quantum feedback networks,” *Physical Review A*, vol. 81, p. 023804, 2010.
- [16] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control*. Cambridge University Press, 2010.
- [17] I. R. Petersen, “Quantum linear systems theory,” in *Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems*, Budapest, Hungary, July 2010.
- [18] M. James and J. Gough, “Quantum dissipative systems and feedback control design by interconnection,” *IEEE Transactions on Automatic Control*, vol. 55, no. 8, pp. 1806 –1821, August 2010.
- [19] C. Gardiner and P. Zoller, *Quantum Noise*. Berlin: Springer, 2000.